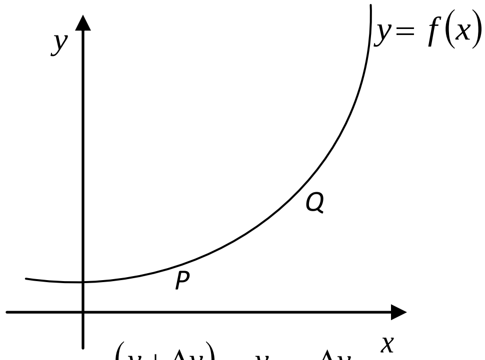
S5 CALCULUS - DIFFERENTIATION

Consider the points $P(x\ ,\ y)$ and $Q(x+\Delta x\ ,\ y+\Delta y)$, very close together on a curve $y=f(x)$, where Δx and Δy are small changes in x and y respectively.



$$Grad\ PQ = \frac{(y+\Delta y)-y}{(x+\Delta x)-x} = \frac{\Delta y}{\Delta x}$$

The gradient function of the curve at the point $P(x\ ,\ y)$ is obtained by taking the point Q move so close to the point P. This gives the derivative of the function $y=f(x)$ at $P(x\ ,\ y)$.

Thus $\frac{dy}{dx} = \lim_{\Delta x \to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0}\left[\frac{f(x+\Delta x)-f(x)}{\Delta x}\right]$…….. (*)

Differentiation from first principles

We shall illustrate this using some examples.

Find the derivatives of the following functions from first principles.

(a) $y = 2x+3$

Let Δx and Δy be small changes in x and y respectively.

$$\Delta y = f(x+\Delta x)-f(x) = 2(x+\Delta x)+3-(2x+3) = 2\Delta x$$

$$\frac{\Delta y}{\Delta x} = \frac{2\Delta x}{\Delta x} = 2$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0}\frac{\Delta y}{\Delta x} = 2$$

(b) $y = x^2$

Let Δx and Δy be small changes in x and y respectively.

$$\Delta y = f(x+\Delta x)-f(x) = (x+\Delta x)^2 - x^2 = x^2 + 2x\Delta x + (\Delta x)^2 = \Delta x(2x+\Delta x)$$

$$\frac{\Delta y}{\Delta x} = \frac{\Delta x(2x + \Delta x)}{\Delta x} = 2x + \Delta x$$

$\frac{dy}{dx} = \lim_{\Delta x \to 0} 2x + \Delta x = 2x + 0 = 2x$ (this is got by substituting Δx with 0)

(c) $y = x^3 - 3$

Let Δx and Δy be small changes in x and y respectively.

$$\Delta y = f(x + \Delta x) - f(x) = (x + \Delta x)^3 - 3 - (x^3 - 3)$$

$$= x^3 + 3x^3 \Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - 3 - (x^3 - 3)$$

$$= \Delta x\left(3x^2 + 3x\Delta x + (\Delta x)^2\right)$$

$$\frac{\Delta y}{\Delta x} = \frac{\Delta x\left(3x^2 + 3x\Delta x + (\Delta x)^2\right)}{\Delta x} = 3x^2 + 3x\Delta x + (\Delta x)^2$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} 3x^2 + 3x\Delta x + (\Delta x)^2 = 3x^2 + 3x(0) + (0)^2 = 3x^2$$

(d) $y = \frac{1}{x}$

Let Δx and Δy be small changes in x and y respectively.

$$\Delta y = f(x + \Delta x) - f(x) = \frac{1}{x + \Delta x} - \frac{1}{x} = \frac{x - x - \Delta x}{x(x + \Delta x)} = -\frac{\Delta x}{x(x + \Delta x)}$$

$$\frac{\Delta y}{\Delta x} = -\frac{\Delta x}{x(x + \Delta x)} \times \frac{1}{\Delta x} = -\frac{1}{x(x + \Delta x)}$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0}\left(-\frac{1}{x(x + \Delta x)}\right) = -\frac{1}{x^2}$$

(e) $y = \frac{1}{x^2}$

Let Δx and Δy be small changes in x and y respectively.

$$\Delta y = f(x + \Delta x) - f(x) = \frac{1}{(x + \Delta x)^2} - \frac{1}{x^2} = \frac{x^2 - (x + \Delta x)^2}{x^2 (x + \Delta x)^2}$$

$$= \frac{x^2 - \left(x^2 + 2x\Delta x + (\Delta x)^2\right)}{x^2 (x + \Delta x)^2} = -\frac{\Delta x(2x + \Delta x)}{x^2 (x + \Delta x)^2}$$

$$\frac{\Delta y}{\Delta x} = -\frac{\Delta x(2x+\Delta x)}{x^2(x+\Delta x)^2} \times \frac{1}{\Delta x} = -\frac{(2x+\Delta x)}{x^2(x+\Delta x)^2}$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0}\left(-\frac{(2x+\Delta x)}{x^2(x+\Delta x)^2}\right) = -\frac{2}{x^3}$$

(f) $y = \sqrt{x}$

Let Δx and Δy be small changes in x and y respectively.

$$\Delta y = f(x+\Delta x) - f(x) = \frac{\sqrt{x+\Delta x} - \sqrt{x}}{1} \quad \text{........ (**)}$$

Here multiply top and bottom of equation (**) by the conjugate of $\sqrt{x+\Delta x} - \sqrt{x}$.

$$\Delta y = \frac{\left(\sqrt{x+\Delta x} - \sqrt{x}\right)\left(\sqrt{x+\Delta x} + \sqrt{x}\right)}{\sqrt{x+\Delta x} + \sqrt{x}} = \frac{x+\Delta x - x}{\sqrt{x+\Delta x} + \sqrt{x}} = \frac{\Delta x}{\sqrt{x+\Delta x} + \sqrt{x}}$$

$$\frac{\Delta y}{\Delta x} = \frac{\Delta x}{\sqrt{x+\Delta x} + \sqrt{x}} \times \frac{1}{\Delta x} = \frac{1}{\sqrt{x+\Delta x} + \sqrt{x}}$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0}\left(\frac{1}{\sqrt{x+\Delta x} + \sqrt{x}}\right) = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

(g) $y = \frac{1}{2\sqrt{x}}$

Let Δx and Δy be small changes in x and y respectively.

$$\Delta y = f(x+\Delta x) - f(x) = \frac{1}{2\sqrt{x+\Delta x}} - \frac{1}{2\sqrt{x}} = \frac{2\sqrt{x} - 2\sqrt{x+\Delta x}}{2\sqrt{x} \cdot 2\sqrt{x+\Delta x}} \quad \text{........ (***)}$$

Here multiply top and bottom of equation (***) by the conjugate of $\sqrt{x} - \sqrt{x+\Delta x}$.

$$\Delta y = \frac{2\sqrt{x} - 2\sqrt{x+\Delta x}}{2\sqrt{x} \cdot 2\sqrt{x+\Delta x}} = \frac{\left(\sqrt{x} - \sqrt{x+\Delta x}\right)\left(\sqrt{x} + \sqrt{x+\Delta x}\right)}{2\sqrt{x(x+\Delta x)}\left(\sqrt{x} + \sqrt{x+\Delta x}\right)} = \frac{x - x - \Delta x}{2\sqrt{x(x+\Delta x)}\left(\sqrt{x} + \sqrt{x+\Delta x}\right)}$$

$$= \frac{-\Delta x}{2\sqrt{x(x+\Delta x)}\left(\sqrt{x} + \sqrt{x+\Delta x}\right)}$$

$$\frac{\Delta y}{\Delta x} = \frac{-\Delta x}{2\sqrt{x(x+\Delta x)}\left(\sqrt{x} + \sqrt{x+\Delta x}\right)} \times \frac{1}{\Delta x} = \frac{-1}{2\sqrt{x(x+\Delta x)}\left(\sqrt{x} + \sqrt{x+\Delta x}\right)}$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0}\left(\frac{-1}{2\sqrt{x(x+\Delta x)}\left(\sqrt{x} + \sqrt{x+\Delta x}\right)}\right) = \frac{-1}{2\sqrt{x(x+0)}\left(\sqrt{x} + \sqrt{x+0}\right)}$$

$$= -\frac{1}{2\sqrt{x^2}\left(2\sqrt{x}\right)} = -\frac{1}{4x^{\frac{3}{2}}}$$

Note:

(i) In all cases, Δy is a multiple of Δx.

(ii) In examples (d), (e) and (g) above, you do not need to expand the denominator when obtaining Δy.

(iii) $\frac{dy}{dx}$ is termed as the gradient function of $y = f(x)$ or it is the first derivative of $y = f(x)$ with respect to x.

ACTIVITY I

Differentiate the following from first principles.

(a) $y = 3 - x$ (b) $y = x^2 + 2$ (c) $y = x^2 + 5x$

(d) $y = 2 - x^2$ (e) $y = x + x^3$ (f) $y = 2\sqrt{x}$

(g) $y = \frac{3}{3+x}$ (h) $y = \frac{1}{x^2+1}$ (i) $y = \frac{1}{1-x}$

(j) $y = \frac{1}{1-x^2}$ (k) $y = \frac{x}{1+x^2}$ (l) $y = \frac{2x}{1-x}$

(m) $y = \frac{1}{2+\sqrt{x}}$ (n) $y = x^3 - 2x + 5$

The rule for differentiation

(a) Suppose that $y = x^n$, then $\frac{dy}{dx} = n\,x^{n-1}$; that is to say "multiply by the power and reduce the power by 1"

Example

Find $\frac{dy}{dx}$ in each of the cases below:

(i) $y = x^2$; $\frac{dy}{dx} = 2x^{2-1} = 2x$

(ii) $y = x^7$; $\frac{dy}{dx} = 7x^{7-1} = 7x^6$

(iii) $y = x^{-1}$; $\frac{dy}{dx} = -x^{-1-1} = -x^{-2} = -\frac{1}{x^2}$

(iv) $y = \frac{1}{x^3} = x^{-3}$; $\frac{dy}{dx} = -3x^{-3-1} = -3x^{-4} = -\frac{3}{x^4}$

(v) $y = x^{\frac{1}{2}}$; $\frac{dy}{dx} = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}}$

(vi) $y = \frac{1}{x^{\frac{3}{2}}} = x^{-\frac{3}{2}}$; $\frac{dy}{dx} = -\frac{3}{2}x^{-\frac{3}{2}-1} = -\frac{3}{2}x^{-\frac{5}{2}}$

(vii) $y = -4x^5$; $\frac{dy}{dx} = -20x^{5-1} = -20x^4$

(b) Given that $y = k$ (a constant), then $\frac{dy}{dx} = 0$.

Proof:

For $y = k = k\,x^0$

Applying the rule from above, $\frac{dy}{dx} = 0 \times k\,x^{0-1} = 0$.

For example, if $y = -3$, then $\frac{dy}{dx} = 0$.

Example

1. Find $\frac{dy}{dx}$ in each of the following cases;

(a) $y = 2x^2 - 3$, $\frac{dy}{dx} = 4x - 0 = 4x$.

(b) $y = 1 - x^4$, $\frac{dy}{dx} = 0 - 4x^3 = -4x^3$.

(c) $y = x^3 - 3x^2 + 5x - 2$, $\frac{dy}{dx} = 3x^2 - 6x + 5$.

(d) $y = 5x + \frac{1}{x^2}$, $\frac{dy}{dx} = 5 - \frac{2}{x^3}$.

2. Find the value of $\frac{dy}{dx}$ fr the following curves at the given points.

(a) $y = 2x^2 - 3x + 4$; $(1, 3)$

$$\frac{dy}{dx} = 4x - 3$$

At $(1, 3)$, $\frac{dy}{dx} = 4 \times 1 - 3 = 1$

(b) $y = x^2 - \frac{1}{x}$; $(1, 0)$

$$\frac{dy}{dx} = 2x + \frac{1}{x^2}$$

At $(1, 0)$, $\frac{dy}{dx} = 2 \times 1 + \frac{1}{1^2} = 3$

3. Determine the values of x for which $\frac{dy}{dx} = 0$.

(a) $y = x^3 - 2x^2 + 4$

$$\frac{dy}{dx} = 3x^2 - 4x = 0$$

$$x(3x - 4) = 0$$

$x = 0$ or $x = \frac{4}{3}$

(b) $y = \frac{4}{3}x^3 - x + 5$

$$\frac{dy}{dx} = 4x^2 - 1 = 0$$

$$(2x - 1)(2x + 1) = 0$$

$$x = \pm\frac{1}{2}$$

(c) $y = 2x + \frac{1}{x}$

$$\frac{dy}{dx} = 2 - \frac{1}{x^2} = 0$$

$$2x^2 - 1 = 0$$

$$x = \pm\frac{\sqrt{2}}{2}$$

ACTIVITY II

1. Determine the values of $\frac{dy}{dx}$ to the curves below at the given – values.

(a) $y = x^4 - 2x + 3$, $x=1$

(b) $y = 3x^2 + 3x - 4$, $x=2$

(c) $y = 1 - x^3$, $x=-1$

(d) $y = x(x-1)(x+1)$, $x=0$

(e) $y = 5 - 2x - x^2$, $x=-1$

(f) $y = (1 + x)^2$, $x=1$

(g) $y = 1 - \frac{1}{x^2}$, $x=-1$

(h) $y = x^3 - 2x^2 - 4$, $x=2$

2. Find the value of the gradient function to the curve at the given value of *x*.

(a) $y = x - \sqrt{x}$, $x=4$

(b) $y = 2\sqrt{x} - \frac{1}{\sqrt{x}}$, $x=1$

(c) $y = x^2 - 4x + 3$, $x=0$

(d) $y = (1-x)(x^2 + 3)$, $x=2$