APPLICATION OF DIFFERENTIATION

RATE OF CHANGE

The derivative $\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$ is called the rate of change of y with respect to x.

It shows how changes in y are related to changes in x. For example, if $\frac{dy}{dx} = 3$, then y is increasing 3 times as fast as x.

Practical application involves;

- The rate of change of displacement (s) with time t $\frac{ds}{dt}$ which is velocity (v).
- The rate of change of velocity (v) with the time t $\frac{dv}{dt}$ which is acceleration (a).

RELATED RATES OF CHANGE

This is where the rate of change in one quantity effects the change in other variables. For example, suppose the side of a square piece of metal increases at a rate of 0.1cm per second when heated. As a result, the area of the square of the metal also increases as well as its volume. Using related rate of change we can fine the rate of increase of area as follows. Let the square piece of metal have sides of length $x$cm,then area A=x^2.

$\frac{dx}{dt}$ Is the rate of increase of length of sides with respect to t. If $A = x^2$ then $\frac{dA}{dx} = 2x$.

The rate of change of area $\frac{dA}{dt}$ is obtained by chain rule as $\frac{dA}{dt} = \frac{dA}{dx} . \frac{dx}{dt} = 2x \times 0.1$

$\therefore \frac{dA}{dx} = 0.2cm^2s^{-1}$

Example 1

A spherical balloon is inflated at a rate of $3cm^3s^{-1}$. Find the rate of increase of the radius when this radius is 2cm.

Solution

Let the balloon have radius r and volume v,

$$v = \frac{4}{3}\pi r^3$$

$$\frac{dv}{dr} = 4\pi r^2$$

$$\frac{dv}{dt} = 3(Given)$$

By chain rule $\frac{dv}{dt} = \frac{dv}{dr}.\frac{dr}{dt}$

$\longrightarrow$ $3 = 4\pi r^2 \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{3}{4\pi r^2}$ when $r = 2$

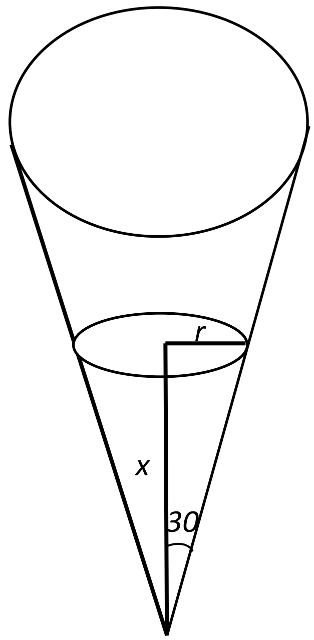
$$\frac{dr}{dt} = \frac{3}{4\pi(2)^2} = \frac{3}{16\pi}$$

Therefore the rate of increase of radius when the radius is 2cm is $\frac{3}{6\pi}cms^{-1}$.

Example 2

A container in the shape of a hallow cone of semi-vertical angle 30^0 is held with its vertex pointing down wards. Water is poured into the cone at the rate of $5cm^3s^{-1}$. Find the rate at which the depth of water in the cone is increasing when this depth is 10cm.

Soln



Let the depth of water in the cone be $x cm$ and radius r cm of the cross-section of the water.

$\tan 30 = \frac{r}{x}$

$r = x\tan 30 = \frac{x}{\sqrt{3}}$

The volume v=$\frac{1}{3}\pi r^2 x = \frac{1}{3}\pi\left(\frac{1}{\sqrt{3}}\right)^2 x = \frac{1}{9}\pi x^3$

$\frac{dv}{dx} = \frac{1}{3}\pi x^2$

Given that $\frac{dv}{dt} = 5$. By chain rule, $\frac{dv}{dt} = \frac{dv}{dx}.\frac{dx}{dt}$

$5 = \frac{1}{3}\pi x^2 \frac{dx}{dt}$

$\frac{dx}{dt} = \frac{15}{\pi x^2}$

When $x = 10$

$$\frac{dx}{dt} = \frac{15}{\pi(10)^2} = \frac{3}{20\pi}$$

The rate of increase of depth when the depth is 10cm = $\frac{3}{20\pi} cm^{-1}$.

Example 3

The radius of a sphere is r cm, the area of its surface is $4\pi r^2 cm^2$ and its volume is $\frac{4}{3}\pi r^3 cm^3$. When the radius of the sphere is 21cm, the radius is increasing at the rate of $0.01cm/s$, find the rate at which the surface area and the volume are increasing at this point. (Given that $\pi = \frac{22}{7}$).

<u>Solution</u>

$$\frac{dr}{dt} = 0.01cm/s, \quad s = 4\pi r^2, \quad v = \frac{4}{3}\pi r^3$$

$$\frac{ds}{dt} = 8\pi r, \quad \frac{ds}{dt} = \frac{ds}{dr}.\frac{dr}{dt}$$

$$\frac{ds}{dt} = 8\pi r(0.01) = 8 \times \frac{22}{7} \times 21(0.01)$$

$\frac{ds}{drt} = 5.28cm^2/s$ (rate of increase of area)

$$\frac{dv}{dt} = \frac{dr}{dt}.\frac{dv}{dr}$$

$$\frac{dv}{dr} = 4\pi r^2$$

$$\Rightarrow \frac{dv}{dt} = 4\pi r^2(0.01)$$

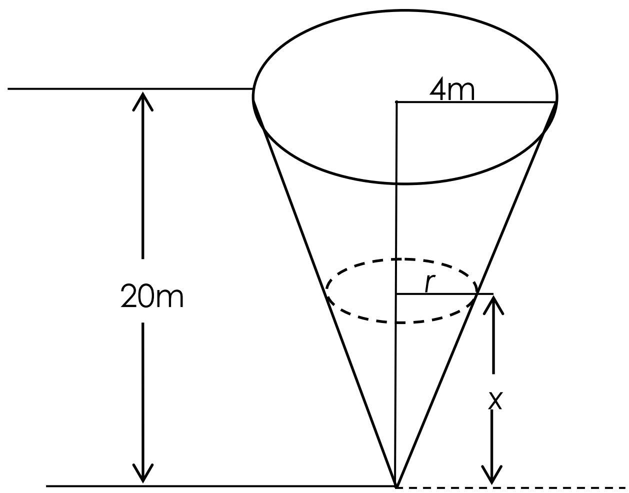
$$= 4 \times \frac{22}{7}(21)^2 \times 0.01$$

$$\frac{dv}{dt} = 55.4cm^3/\sec.$$

Example 4

A hollow right circular cone has base radius 4m and vertical height 20m. It is held upside down with its axis vertical. It contains water which is being added at the constant rate of 1.5cm³ per minute and which leaks away through a small hole in vertex at the constant rate of 2m³ per minute. At what rate is the depth of the water changing when the depth is 12m?

Solution



By similarity: $\frac{x}{20} = \frac{r}{4} \Rightarrow r = x/5$ volume of water of radius r and height x

$$v = \frac{1}{3}\pi r^2 h,\ h = x$$

$$= \frac{1}{3}\pi\left(\frac{x}{5}\right)^2 x = \frac{1}{3}\pi\frac{x^2}{25}x$$

$$vol. = \frac{\pi x^3}{75}$$

$$\frac{dv}{dt} = \frac{3\pi x^2}{75} = \frac{\pi x^2}{25}$$

$\frac{dv}{dt} = 1.5 - 2 = -0.5m^3/min$ .The negative indicates decrease in volume.

$$\frac{dv}{dt} = \frac{dx}{dt}.\frac{dv}{dx}$$

$$\frac{dx}{dt} = \frac{dv}{dt}.\frac{dx}{dv}, \quad \frac{dx}{dv} = \frac{25}{\pi x^2}$$

$$= -0.5.\frac{25}{\pi(x)^2} \qquad x = 2$$

$$= -0.5.\frac{25}{\pi(x)^2} \qquad x = 12$$

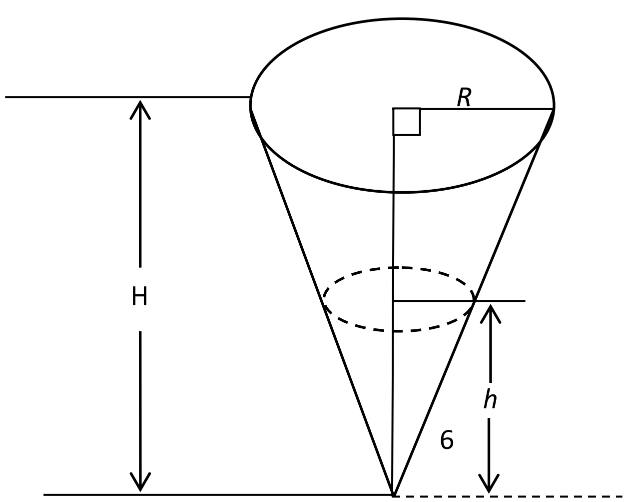
$$= -0.5 \times \frac{25 \times 7}{22 \times (12)^2}$$

$$= \frac{dx}{dt} = 0.03m/\min(2dp)$$

Example 5

A large container in the shape of a right circular cone of height 10m and base radius 1m is catching a drip of water from a tap leaking at the rate of 0.1m³/min. Find the rate at which the area of the top surface of water is increasing when water is half way up the cone.

Solution



Let the height of water at any time t be h.

By similarity $\frac{h}{H} = \frac{r}{R}$

= $\frac{h}{10} = \frac{r}{1} \Rightarrow r = h/10$

Area of the top surface of water = πr^2

➡ $A = \pi \left(h/10\right)^2 = \frac{\pi h^2}{100}$

$\frac{dA}{dh} = \frac{\pi h}{5^0}$

$\frac{dA}{dt} = \frac{dA}{dh} \cdot \frac{dh}{dt}$

$\frac{dA}{dt} = \frac{\pi h}{50} \frac{dh}{dt}$ ———— (1)

Volume $v = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \frac{h^2}{100} h$

$v = \frac{\pi h^3}{300}$

$\frac{dv}{dh} = \frac{\pi h^2}{100}$

$\frac{dv}{dt} = \frac{dv}{dh} \cdot \frac{dh}{dt}$. From chain rule but $\frac{dv}{dt} = 0.1 m^3 / \min$

$\Rightarrow 0.1 = \frac{\pi h^2}{100} \frac{dh}{dt}$

$\Rightarrow \frac{dh}{dt} = \frac{100 \times 0.1}{\pi h^2} = \frac{10}{\pi h^2}$

Substituting for $\frac{dh}{dt}$ in (1)

$\frac{dA}{dt} = \frac{1}{5h}$ .When $h = \frac{10}{2} = 5m$

$\frac{dA}{dt} = \frac{1}{5 \times 5} = \frac{1}{25} m^2$ $\qquad$ $\frac{dA}{dt} = 0.04 m^2 / \text{min}$

Example 6

A spherical balloon is inflated such that the rate at which its radius is increasing is $0.5cms^{-1}$. Find the rate at which;

i) The volume is increasing at the point when $r = 5.0cm$
ii) Its surface area is increasing when $r = 5.0cm$

(i) $v = \frac{4}{3}\pi r^3$, $\frac{dv}{dr} = 4\pi r^2$

$\frac{dr}{dt} = 0.5cms^{-1}$

$\frac{dv}{dt} = \frac{dv}{dr} - \frac{dr}{dt}$

$= \frac{dv}{dt} = 4\pi r^2 (0.5)$

$= 2\pi r^2$ when $r = 5$

$\frac{dv}{dt} = 2\pi (5)^2 = 50\pi cm^3 s^{-1}$

(ii) $S = 4\pi r^2$, $\frac{ds}{dr} = 8\pi r$

$\frac{ds}{dt} = \frac{ds}{dr} . \frac{dr}{dt} = 8\pi r.0.5$ (Using chain rule).

$= 4\pi r$, at $r = 5$

$$\frac{ds}{dt}=4\pi(5)=20\pi cm^2s^{-1}$$

Example 5

Spherical rain drop is formed by condensation. In an interval of 40 seconds its volume increases at a constant rate from 0.032mm³ to 0.256mm³. Find the rate at which the surface area of the rain drop increasing when its radius is 0.5mm.

Solution

Volume V of the spherical rain drop is

$$v=\frac{4}{3}\pi r^3,$$

$$\frac{dv}{dr}=4\pi r^2$$

$\frac{dv}{dt}=\frac{dv}{dr}.\frac{dr}{dt}$, (Using chain rule). But $\frac{dv}{dt}=\frac{0.256-0.032}{40}$

$$\frac{dv}{dt}=0.0056mm^3/s$$

$\Rightarrow \quad 0.0056=4\pi r^2.\frac{dr}{dt}$

$$\frac{dr}{dt}=\frac{0.0056}{4\pi r^2} \quad \text{———— (1)}$$

Surface area $A=4\pi r^2 \quad \frac{dA}{dr}=8\pi r$

By chain rule, $\frac{dA}{dt} = \frac{dA}{dr}.\frac{dr}{dt} = 8\pi r \frac{0.0056}{4\pi r^2} = \frac{2}{r} \times 0.0056$. $\frac{dr}{dt}$ is obtained equation 1 above.

Therefore , $\frac{dA}{dt} = 0.0224 mm^2 / s$.

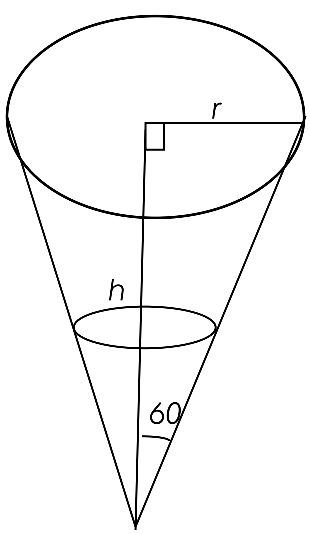
Example 8

A inverted cone of vertical angle 120^0 is collecting water from a tap at a steady rate of 18π cm^3/min. Find;

(a) the depth of water after 12 min.

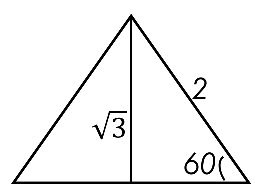
(b) the rate of increase of depth at this instant

Solution



$tan60 = \frac{r}{h}$

r = h tan 60



$$v = \frac{1}{3}\pi r^2 h$$

$$v = \frac{1}{3}\left(h\sqrt{3}\right)^2 h$$

$$v = \frac{1}{3}\pi 3h^2.h = \frac{1}{3}\pi r$$

$$v = \pi h^3$$ This is the

volume of water at any time t.

$$\frac{dv}{dt} = 18\pi cm.^3 / m$$

$$\frac{dh}{dt} = \frac{dv}{dt}.\frac{dh}{dh} = 18\pi.\frac{1}{3\pi h^2}$$

$$\frac{dv}{dh} = 3\pi h^2$$

$$\frac{18}{3h^2} = \frac{6}{h^2}$$

$$\frac{dh}{dt} = \frac{6}{h^2}$$

1min=18πcm^3

12min=12x18πcm^3=216πcm^3.

Volume of water after 12 min =216πcm^3.

Also volume of water =πh$^{3.}$ (From above).

Therefore 216π=πh^3

h^3= 216

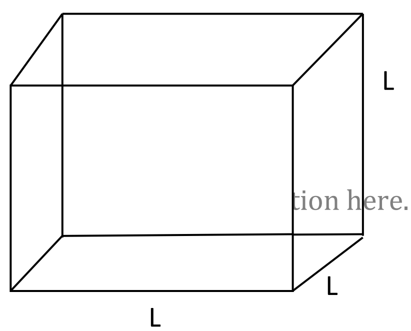
h = 6

$\frac{dh}{dt} = \frac{6}{6^2} = \frac{1}{6}$cm^2/min

Example 9

A piece of metal is the shape of a cube of side 10cm is heated uniformly so that its volume increases at a rate proportional to the area of one face. It is known that when one of the sides is 10.5m, the rate of increase of the volume is 11.25m^3/min. Find the rate at which the length L metre of each edge increases.

Solution



Let V be the volume of metal cube and S the surface area.

$v = L^3$ and $S = L^2$

$\frac{dv}{dt} = kL^2 (given)$

at $L = 10.5m \frac{dv}{dt} = 11.025 (given)$

but $\frac{dv}{dt} = kL^2$

$\Rightarrow kL^2 = 11.025$

$K = \frac{11.025}{L^2} = \frac{11.025}{(10.5)^2} = 0.1$

$\Rightarrow \frac{dv}{dt} = 0.1L^2$

$\frac{dv}{dt} = \frac{dv}{dt} . \frac{dL}{dt}$

$$\frac{dL}{dt}=\frac{dv}{dt}.\frac{dL}{dv}=\frac{dv}{dt}\times\frac{1}{\frac{dv}{dL}}$$

$$=0.1L^2.\frac{1}{3L^2}=\frac{0.1}{3}$$

$$=0.03m/\min$$

Hence the rate at which the length of each edge increases is 0.03m/min.

EXERCISE

1. The side of a square is increasing at the rate of $5cm^{-1}$. Find the rate of Increase of the area when the length of the side is 10cm.
   Ans. $100cm^2s^{-1}$

2. The area of a circle is increasing at the rate of $(4\pi)cm^2s^{-1}$. Find the rate of increase of the radius when radius is $\frac{1}{2}cm$.
   Ans. $4cms^{-1}$

3. The volume of cube is increasing at the rate of 18cm3s^{-1}. Find the rate of increase of the length of a side when the volume is $125cm^3$.
   Ans. $\frac{6}{25}cms^{-1}$

4. The area of a square, of side xcm, is increasing at the rate of $6cm^2s^{-1}$. Find the expression interim of x, for the rate of increase of the length of a side. Ans. $\frac{3}{x}cms^{-1}$

5. A boy is inflating a spherical balloon at the rate of $10cm^3s^{-1}$. Find the rate of increase of surface area of the balloon when the radius is 5cm.
   Ans. $4cm^2s^{-1}$

6. A container in the shape of a hallow cone of semi-vertical angle 45^0 is held with its vertex pointing down wards. Water drips into the container at a rate of $3cm^3$ per minute. Find the rate at which the depth of water in the cone is increasing when the depth is 2cm.
   Ans. $\frac{3}{4\pi}cm\min^{-1}$

7. A container in the shape of a hallow cone of depth 12cm and radius 6cm

is held with its vertex pointing down wards. Water is poured into the container at a rate of 2cm^{3}s^{-1}. Find the rate at which the depth of water in the cone is increasing when the depth is $\frac{3}{4}cm$.

Ans. $\frac{128}{9\pi}cms^{-1}$

8. A hallow cone of base radius 10cm and height 10cm is held with its vertex down wards. The cone is initially empty when water is poured into it at the rate of (4π)cm^{3}s^{-1}. Find the rate of increase in the depth of water in the cone 18 seconds after pouring has commenced.

Ans. $\frac{1}{9}cms^{-1}$

9. Sand pouring at a constant rate of $\left(\frac{1000}{3}\pi\right)cm^3s^{-1}$ forms a right circular conical heap of height 10cm. Express the volume of heap interims of the vertical angle. hence, find the rate of change of vertical angle when its 60^0.

Ans. $v=\frac{1}{3}h^3\pi\tan^2\theta$, $\frac{d\theta}{dt}=\left(\frac{3\sqrt{3}}{8}\right)s^{-1}$

10. An inverted cone with vertical angle 60^0 has water in it dripping out through a hole at the vertex at a rate of 3cm^3 min^{-1}. Find the rate at which the surface area in contact with water is changing at the instant when the volume left in the cone is $8\pi cm^3$.

Ans. 1.44cm^{2}min^{-1}

11. Water runs at a constant rate of 6cm^{2}s^{-1} in vessel whose volume is obtained by rotating the area bounded by the curve $4y=x^2$ about the y-axis from $y=0$ to $y=hcm$.
    (i) Show that the volume of the vessel is $2\pi h^2cm^3$.
    (ii) Find the rate at which the water level is rising when the water has been running for 3 second.

Ans (ii) $\frac{1}{2\sqrt{\pi}} cms^{-1}$

12. A hemispherical bowel of radius r cm is being filled with water at a Constant rate.

(a) Show that when the depth of water in the bowl is hcm, then the volume of water in the bowl is given by $v = \frac{\pi h^2(3r-h)}{3}$

(b) Show that between the time water is half way to the top and the time when the bowl is about to overflow, The rate at which the depth is rising has fallen by a quarter.

END